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MATH 730 Final

1. Let G, H be topological groups that are path-connected and semi-locally 1-connected.

(a) Assume that N is a discrete normal subgroup of G. Show that N lies in the center of G.

For a topological group, the group action is a continuous function. Therefore, for every $n \in N$ the function $f_n(g) = gng^{-1}n^{-1}$ is a continuous function from $G \to N$ (since N is normal). As N is a discrete space and G is path-connected and thus connected, f_n is constant. If we let e denote the identity element of G, then $f_n(e) = e$ for every n. Therefore $gng^{-1}n^{-1} = e$ for every $g \in G$ and every $n \in N$. Therefore $N \subset Z(G)$.

(b) Let $f : G \to H$ be a homomorphism which is also a covering map. Show the kernel of f is abelian.

Since f is a homomorphism $\mathrm{Ker}(f) \lhd G$. Also, f is a covering map, so $\mathrm{Ker}(f) = f^{-1}(e)$ is a discrete set of points (as it is the fiber at e). Therefore $\mathrm{Ker}(f)$ is a discrete normal subgroup of G. So by part (a), $\mathrm{Ker}(f) \subset Z(G)$. Thus if $a \in \mathrm{Ker}(f)$ and $b \in \mathrm{Ker}(f) \subset G$ then $bab^{-1}a^{-1} = e \Rightarrow ba = ab$. Therefore $\mathrm{Ker}(f)$ is abelian.

(c) Show that the fundamental group of H is abelian.

Let f and g be arbitrary elements of $\pi_1(H,e)$. Then $f, g : (I, \partial I) \to (G,e)$. Let $f * g$ represent the group action in $\pi_1(H,e)$, i.e.

$$f * g(t) = \begin{cases} f(2t) & 0 \le t \le 1/2 \\ g(2t-1) & 1/2 \le t \le 1. \end{cases}$$

Also, let adjacency represent the group action of H so that $fg(t) = f(t)g(t)$. Then there exists a homotopy $\mathbf{H}_{fg} : f * g \approx fg, \operatorname{rel} \partial I$ given by:

$$\mathbf{H}_{fg}(t,s) = \begin{cases} f\left(\frac{2t}{1+s}\right) & 0 \le t \le \frac{1-s}{2} \\ f\left(\frac{2t}{1+s}\right)g\left(\frac{2t-1+s}{1+s}\right) & \frac{1-s}{2} \le t \le \frac{1+s}{2} \\ g\left(\frac{2t-1+s}{1+s}\right) & \frac{1+s}{2} \le t \le 1 \end{cases}.$$

To verify this, we need to check that it is well defined and satisfies the boundary conditions.

Well-Defined: For $t = \frac{1-s}{2}$ we get

$$\begin{aligned} \mathbf{H}_{fg}(t,s) &= f\left(\frac{2t}{1+s}\right)g\left(\frac{2t-1+s}{1+s}\right) \\ &= f\left(\frac{2t}{1+s}\right)g\left(\frac{1-s-1+s}{1+s}\right) \\ &= f\left(\frac{2t}{1+s}\right)g(0) \\ &= f\left(\frac{2t}{1+s}\right)e \\ &= f\left(\frac{2t}{1+s}\right). \end{aligned}$$

For $t = \frac{1+s}{2}$ we get

$$\begin{aligned} \mathbf{H}_{fg}(t,s) &= f\left(\frac{2t}{1+s}\right)g\left(\frac{2t-1+s}{1+s}\right) \\ &= f\left(\frac{1+s}{1+s}\right)g\left(\frac{2t-1+s}{1+s}\right) \\ &= f(1)g\left(\frac{2t-1+s}{1+s}\right) \\ &= eg\left(\frac{2t-1+s}{1+s}\right) \\ &= g\left(\frac{2t-1+s}{1+s}\right). \end{aligned}$$

Thus our homotopy is well-defined.

Boundary Conditions:

$$\mathbf{H}_{fg}(t,0)=\begin{cases} f(2t) & 0\le t\le 1/2 \\ e & t=1/2 \\ g(2t-1) & 1/2\le t\le 1\end{cases}$$

which equals $f*g(t)$. Since $f(1)=g(0)=e$.

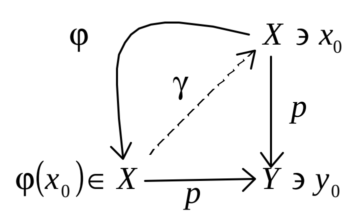
$$\mathbf{H}_{fg}(t,1)=\begin{cases} f(0)=e & t=0 \\ f(t)g(t) & 0\le t\le 1 \\ g(1)=e & t=1\end{cases}$$

which equals $fg(t)$. Finally, on ∂I we have $\mathbf{H}_{fg}(0,s)=f(0)=e$ and $\mathbf{H}_{fg}(1,s)=g(1)=e$ for every s. Therefore $\mathbf{H}_{fg}: f*g\approx fg, \operatorname{rel}\partial I$.

There also exists the homotopy $\tilde{\mathbf{H}}: fg\approx gf, \operatorname{rel}\partial I$ given by $\tilde{\mathbf{H}}(t,s)=[f(st)]^{-1}f(t)g(t)f(st)$. A quick check shows $\tilde{\mathbf{H}}(t,0)=f(t)g(t)$, $\tilde{\mathbf{H}}(t,1)=g(t)f(t)$, $\tilde{\mathbf{H}}(0,s)=e$, and $\tilde{\mathbf{H}}(1,s)=e$ Therefore the homotopy $\mathbf{H}_{gf}^{-1}\tilde{\mathbf{H}}\mathbf{H}_{fg}: f*g\approx g*f, \operatorname{rel}\partial I$ and thus $f*g$ and $g*f$ are in the same homotopy class $\Rightarrow$ $\pi_1(H,e)$ is abelian. Since H is path-connected, the base point is arbitrary and so, in the most general sense, the fundamental group of H is abelian.

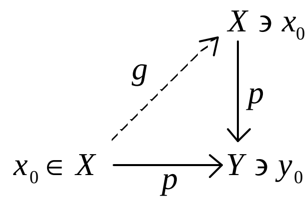
2. If $p: X\to Y$ is a covering map, and $\varphi: X\to X$ is a map such that $p\circ\varphi=p$, then φ is a homeomorphism.

Fix $x_0\in X$ and set $y_0=p(x_0)$. Since $p\circ\varphi=p$, then $p(\varphi(x_0))=y_0$ and $p_\#\pi_1(X,\varphi(x_0))\subset p_\#\pi_1(X,x_0)$ (also note that X is path-connected and locally path-connected since it is a covering space). Therefore, by the Lifting Theorem, there exists a unique continuous map $\gamma:(X,\varphi(x_0))\to(X,x_0)$ such that $p\circ\gamma=p$.

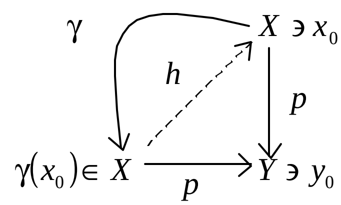


Claim: $\gamma=\varphi^{-1}$.

Since $p_{\#}\pi_1(X,x_0)\subset p_{\#}\pi_1(X,x_0)$, by the Lifting Theorem there exists a unique $g:(X,x_0)\to(X,x_0)$ such that $p\circ g=p$. The obvious choice is $g=id_X$. However, $(\gamma\circ\varphi)(x_0)=\gamma(\varphi(x_0))=x_0$ and $p\circ(\gamma\circ\varphi)=$



$(p\circ\gamma)\circ\varphi=p\circ\varphi=p$. Therefore, by the uniqueness in the Lifting Theorem, $\gamma\circ\varphi=id_X$. Now, since $p\circ\gamma=p$, then $p(\gamma(x_0))=y_0$ and $p_{\#}\pi_1(X,\gamma(x_0))\subset p_{\#}\pi_1(X,x_0)$. Therefore, by the Lifting Theorem, there exists a unique $h:(X,\gamma(x_0))\to(X,x_0)$ such that $p\circ h=p$. But now, again, $(h\circ\gamma)(x_0)=x_0$ and $p\circ(h\circ\gamma)=p$.



Thus, $h\circ\gamma=g=id_X$. Finally, $h=h\circ(\gamma\circ\varphi)=(h\circ\gamma)\circ\varphi=\varphi$. Therefore $h=\varphi$, and $\varphi\circ\gamma=id_X$. Thus $\gamma=\varphi^{-1}$ and is continuous and so φ is a homeomorphism.

3. Let $\gamma:\mathbf{R}\to\mathbf{R}^2$ be a smooth curve in the plane. Let K be the set of all $r\in\mathbf{R}$ such that the circle of radius r (centered at a fixed point) is tangent to γ at some point. Show that K has empty interior.

Without loss of generality, let the center of our circles be the origin. Let $h(t)=\text{dist}_{\mathbf{R}^2}(\gamma(t),(0,0))=\sqrt{(\gamma_1(t))^2+(\gamma_2(t))^2}$ where $(\gamma_1(t),\gamma_2(t))=\gamma(t)$. Since the distance function is a smooth map and so is γ, then $h:\mathbf{R}\to\mathbf{R}$ is a smooth map and $h_*(t)=\dfrac{d}{dt}h(t)=\dfrac{\gamma_1(t)\gamma_1'(t)+\gamma_2(t)\gamma_2'(t)}{h(t)}$. Now suppose that γ was tangent to the circle of radius R at $t=t_0$. Then the vector $(\gamma_1(t_0),\gamma_2(t_0))$ would be perpendicular to the vector $(\gamma_1'(t_0),\gamma_2'(t_0))$. Thus

$$(\gamma_1(t_0),\gamma_2(t_0))\cdot(\gamma_1'(t_0),\gamma_2'(t_0))=\gamma_1(t_0)\gamma_1'(t_0)+\gamma_2(t_0)\gamma_2'(t_0)=0.$$

Therefore $h_*(t_0)=0$ making t_0 a critical point of h and R a critical value. Thus $K\subset\{\text{critical values of } h\}$. Therefore, by Sard's Theorem, K has measure zero which directly implies that K has no interior.

4.(a) Prove that a completely regular space is regular.

Let $x\in X$ and $C\subset X$ closed with $x\notin C$. Then, since X is completely regular, there exists $f:X\to[0,1]$ such that $f(x)=0$ and $f(C)=\{1\}$. Let $U=f^{-1}([0,\tfrac{1}{2}))$ and $V=f^{-1}((\tfrac{1}{2},1])$. Then $x\in U$, $C\subset V$ and $U\cap V=\phi$ since if $y\in U\cap V$ then $\tfrac{1}{2}<f(y)<\tfrac{1}{2}.\Rightarrow\Leftarrow$. Also U and V are open sets since they are the inverses of open sets through a continuous function. Therefore, for every $x\in X$ and $C\subset X$ closed with $x\notin C$, there exists open sets U and V such that $x\in U$, $C\subset V$ and $U\cap V=\phi$. Thus X is regular.

(b) Let X be completely regular, K a compact subspace, and U an open neighborhood of K. Prove that there exists a map $f:X\to[0,1]$ such that $f(K)=\{0\}$ and $f(X-U)=\{1\}$.

Since X is completely regular and $X-U$ is closed, for every $x\in K$ there exists $f_x:X\to[0,1]$ such that $f_x(x)=0$ and $f_x(X-U)=\{1\}$. Let $U_x=f_x^{-1}([0,\tfrac{1}{2}))\cap U$. Since it is the intersection of two open sets, U_x is an open set containing x. Therefore the collection $\{U_x\}_{x\in K}$ is an open cover of K. Since K is compact, there exists a finite subcover $\{U_{x_1},U_{x_2},\ldots,U_{x_n}\}$ such that $K\subset\bigcup_{i=1}^{n}U_{x_i}\subset U$. Now create the continuous function $g(x)=\frac{1}{n}\sum_{i=1}^{n}f_{x_i}(x)$. If $x\in X-U$, then $g(x)=1$. If $x\in K$, then there exists x_j such that $x\in U_{x_j}$ thus

$$
\begin{aligned}
g(x) &= \frac{1}{n}\left(f_{x_j}(x) + \sum_{\substack{1\le i\le n \\ i\ne j}} f_{x_i}(x) \right) \\
&< \frac{1}{n}\left(\tfrac{1}{2} + (n-1) \right) \\
&= 1 - \frac{1}{2n}.
\end{aligned}
$$

Let $m \in \left(1 - \frac{1}{2n}, 1\right)$. Then $g(x) < m$ for every $x \in K$ and $g(x) = 1$ for every $x \in X - U$. Now let $h : \mathbf{R} \to \mathbf{R}$ by setting

$$
h(x) = \begin{cases} 0 & 0 \le x \le m \\ \dfrac{x-m}{1-m} & m \le x \le 1. \end{cases}
$$

h is continuous and thus so is $f = h \circ g : X \to [0,1]$. But now for every $x \in K$, $f(x) = 0$, and for every $x \in X - U$, $f(x) = 1$.

5. Let $p : \mathbf{R}P^n \to X$ be a covering map. What are the possible values of the Euler characteristic $\chi(X)$. Give examples of all possibilities.

S^n is a double cover of $\mathbf{R}P^n$, therefore $\chi(S^n) = 2\chi(\mathbf{R}P^n)$.

Lemma: $\chi(S^n) = 1 + (-1)^n$.

Proof: For $n > 0$, the simplicial complex that minimally represents S^n is a collection of $(n+2)$ vertices such that they do not lie in the same nD-hyperplane along with every possible edge, face, 3-simplex, ... , and $(n+1)$-simplex. Thus, the number of m-simplicies is the binomial coefficient $\binom{n+2}{m+1}$. This yields $\chi(S^n) = \sum_{i=0}^{n} (-1)^i \binom{n+2}{i+1}$ which shall now be shown (via induction) to be equal to $1 + (-1)^n$. The base case is obvious: $\chi(S^1) = 3 - 3 = 0 = 1 + (-1)^1$. Now suppose that $\sum_{i=0}^{n} (-1)^i \binom{n+2}{i+1} = 1 + (-1)^n$ and consider

$$\begin{aligned}\chi(S^{n+1}) &= \sum_{i=0}^{n+1}(-1)^i\binom{n+3}{i+1}\\ &= \sum_{i=0}^{n+1}(-1)^i\left[\binom{n+2}{i}+\binom{n+2}{i+1}\right]\\ &= \sum_{i=0}^{n+1}(-1)^i\binom{n+2}{i}+\sum_{i=0}^{n}(-1)^i\binom{n+2}{i+1}+(-1)^{n+1}\binom{n+2}{n+2}\\ &= \sum_{i=-1}^{n}(-1)^{i+1}\binom{n+2}{i+1}+1+(-1)^n+(-1)^{n+1}\\ &= \binom{n+2}{0}-\sum_{i=0}^{n}(-1)^i\binom{n+2}{i+1}+1\\ &= 1-\left(1+(-1)^n\right)+1\\ &= 1+(-1)^{n+1}\end{aligned}$$

Therefore, $\chi(S^n)=1+(-1)^n$.

Now for n even, $2=\chi(S^n)=2\chi(\mathbf{R}P^n)$, therefore $\chi(\mathbf{R}P^n)=1$. Since $\mathbf{R}P^n$ covers *X*, there exists a positive integer m such that $\mathbf{R}P^n$ is an m-fold cover. Therefore $1=\chi(\mathbf{R}P^n)=m\chi(X)\Rightarrow \chi(X)=1$.

For n odd, $0=\chi(S^n)=2\chi(\mathbf{R}P^n)$, therefore $\chi(\mathbf{R}P^n)=0$. Again, there exists a positive integer m such that $\mathbf{R}P^n$ is an m-fold cover of *X*. Therefore $0=\chi(\mathbf{R}P^n)=m\chi(X)\Rightarrow\chi(X)=0$.

Since $\chi(X)=\chi(\mathbf{R}P^n)$ in both cases, the obvious (though somewhat trivial) example in each is that $\mathbf{R}P^n$ is a 1-fold covering of $X=\mathbf{R}P^n$.

6. Let *K* be a 4-dimensional simplicial complex which has 8 0-simplies, 12 1-simplicies, 9 2-simplicies, 10 3-simplicies, and 6 4-simplices. Suppose that

$$H_0(K)=\mathbf{Z},\ H_1(K)=\mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}_2,\ H_2(K)=\mathbf{Z}\oplus\mathbf{Z}_3,\ H_3(K)=\mathbf{Z}\otimes\mathbf{Z}_4.$$

What is $H_4(K)$?

By definition $H_4(K)=\operatorname{Ker}\partial_4/\operatorname{Im}\partial_5$. Since *K* is 4-dimensional, then $C_5K=0$ which means that $\operatorname{Im}\partial_5=0$. Therefore $H_4(K)=\operatorname{Ker}\partial_4\subset C_4K$ which means H_4K is free. Now, set $n_i = \#\text{ of } i\text{-simplicies}$. Then $\chi(K) = \sum_{i=0}^{n}(-1)^i n_i = 8-12+9-10+6=1$. But also $\chi(K) = \sum_{i=0}^{\infty}(-1)^i \operatorname{rank}(H_iK)$. Therefore $1 = 1-2+1-1+\operatorname{rank}(H_4K) = -1+\operatorname{rank}(H_4K) \Rightarrow \operatorname{rank}(H_4K) = 2$. Thus, $H_4K \cong \mathbf{Z} \oplus \mathbf{Z}$.

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